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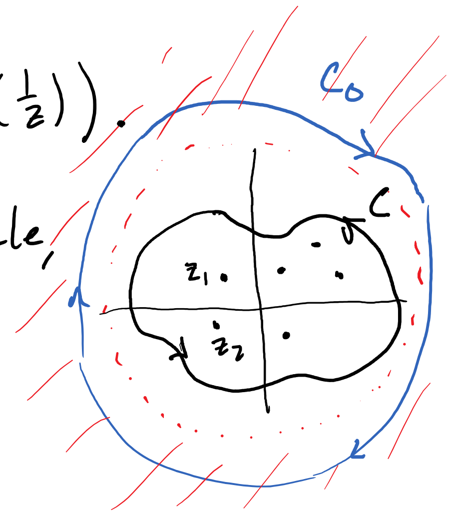
$\boxed{\text{Theorem (Residues at } \infty)}$ If a function is analytic everywhere on $\mathbb{C}$ except at a finite number of singularities lying interior to a simple closed pos. oriented contour C, then

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right).$$

That is,

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right).$$



Proof. Let C_0 be a negatively oriented circle, centered at 0, whose interior contains C. By Principle of Deformation of Paths,

$$\int_C f(z)\,dz = \int_{-C_0} f(z)\,dz$$

$$= -\int_{C_0} f(z)\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z).$$

To compute $\operatorname*{Res}_{z=\infty} f(z)$, we find the Laurent series of f on an annulus $R < |z| < \infty$ where $\max_{w\in C}|w| < R <$ radius C_0. We have

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \frac{1}{z^n}$$

where

$$a_n = \frac{1}{2\pi i}\int_{-C_0} \frac{f(z)}{z^{n+1}}\,dz, \quad b_n = \frac{1}{2\pi i}\int_{-C_0} \frac{f(z)}{z^{-n+1}}\,dz.$$

Then

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n \frac{1}{z^{n+2}} + \sum_{n=1}^{\infty} b_n z^{n-2}, \quad \left(0 < \left|\frac{1}{z}\right| < \frac{1}{R}\right).$$

Note that
$$\operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = b_1 = \frac{1}{2\pi i}\int_{-C_0} f(z)\,dz$$
$$= -\operatorname*{Res}_{z=\infty} f(z).$$

So this proves the claim.

$\blacksquare$

$\boxed{\text{Example}}$ Let C be the circle $|z|=2$ w/ positive orientation. We can compute
$$\int_C \frac{4z-5}{z(z-1)}\,dz$$
using a single residue. Let $f(z) = \frac{4z-5}{z(z-1)}$. This function has singularities at $z=0,1$, both of which lie interior to C. We have
$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\left(\frac{\frac{4}{z}-5}{\frac{1}{z}\left(\frac{1}{z}-1\right)}\right) = \frac{1}{z^2}\left(\frac{\frac{4-5z}{z}}{\frac{1}{z}\left(\frac{1-z}{z}\right)}\right)$$
$$= \cancel{\frac{1}{z^2}}\left(\frac{(4-5z)\cancel{z^2}}{z(1-z)}\right)$$
$$= \frac{4-5z}{z(1-z)}.$$
Notice that $\frac{4-5z}{1-z}$ is analytic at zero so it has a Maclaurin series:
$$(4-5z)\frac{1}{1-z} = (4-5z)\sum_{n=0}^{\infty} z^n = 4 + \underset{\sim\sim\sim}{\text{stuff}}$$

Hence, the coefficient of $\frac{1}{z}$ in $\frac{1}{z}\cdot\frac{4-5z}{1-z}$ is 4.

So $\operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) = 4$ and

$$\int_C \frac{4z-5}{z(1-z)}\,dz = 2\pi i \cdot 4 = 8\pi i$$

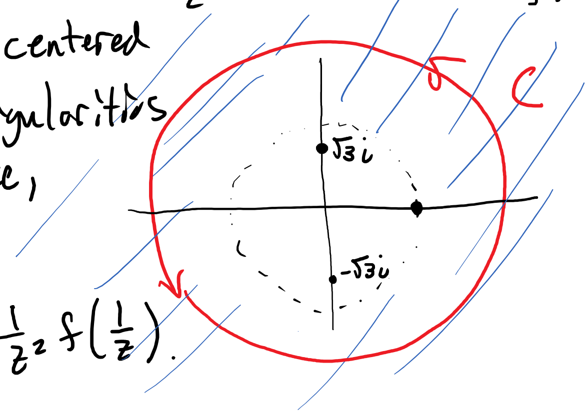
by the theorem. //

$\boxed{\text{Example}}$ The function

Compare w/ Pset 7

$$f(z) = \frac{z^2}{(z-2)(z^2+3)}$$

has no antiderivative on the domain $D = \{z \in \mathbb{C} : |z| > 2\}$. Let C be the circle of radius 3 centered at zero. Notice that all three singularities of f lie interior to C. Hence, by the theorem,



$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right).$$

We have

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2}\left(\frac{\frac{1}{z^2}}{\left(\frac{1-2z}{z}\right)\left(\frac{1+3z^2}{z^2}\right)}\right) = \frac{1}{z^2}\left(\frac{z^3}{z^2(1-2z)(1+3z^2)}\right)$$

$$= \frac{1}{z(1-2z)(1+3z^2)}.$$

The functions $\frac{1}{1-2z}$ and $\frac{1}{1+3z^2}$ are analytic at $z=0$, hence

$$\frac{1}{1-2z} = \sum_{n=0}^{\infty} (2z)^n \quad \text{and} \quad \frac{1}{1+3z^2} = \sum_{n=0}^{\infty} (-1)^n (3z^2)^n$$

and so the Taylor series of the product $\left(\frac{1}{1-2z}\right)\left(\frac{1}{1+3z^2}\right)$ has constant term equal to 1 and hence

$$\operatorname*{Res}_{z=0} \frac{1}{z(1-2z)(1+3z^2)} = 1.$$

Hence, $\int_C f(z)\,dz = 2\pi i$ and by the fund. thm. of contour integrals, f has no antiderivative on D. //

Classifying Isolated Singularities

Recall: If f has an isolated singularity at $z_0 \in \mathbb{C}$, then f has a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n}$$

on some annulus $0 < |z-z_0| < R$. The sum

$$\sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n} = \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \cdots$$

is called the principal part of f at z_0. We will classify isolated singularities based on whether the principal part is zero, nonzero w/ finitely many terms, or nonzero w/ infinitely many terms. The goal is to understand how to compute residues based on the type of singularity

Definition (Types of Singularities) Suppose that f has an isolated singularity at $z_0 \in \mathbb{C}$.

(1) z_0 is an removable singularity if the principal part of f at z_0 is zero, that is, $b_n = 0 \ \forall n \geq 1$.

(2) z_0 is an essential singularity if the principal

part of f at z_0 has infinitely many nonzero terms.

(3) z_0 is a pole of order m if there exists $m \geq 1$ such that $b_m \neq 0$ and $b_n = 0$ for all $n > m$. In this case, the principal part is of the form

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}.$$

A pole of order $m=1$ is called a simple pole. //

Remark (Removable Singularities) Suppose $z_0 \in \mathbb{C}$ is a removable singularity. By definition, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + 0$$

on some annulus $0 < |z-z_0| < R$. If we define

$$g(z) = \begin{cases} f(z), & z \neq z_0 \\ a_0, & z = z_0 \end{cases}$$

then $g(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ on the disk $|z-z_0| < R$.

Hence, $g(z)$ is analytic on the disk and agrees w/ f everywhere on the annulus $0 < |z-z_0| < R$. So, the singularity has been removed. //

Example

(1) (c.f. Pset 6 P4) You proved that

$$g(z) = \begin{cases} \frac{1-\cos z}{z^2}, & z \neq 0 \\ \frac{1}{2}, & z = 0 \end{cases}$$

is entire. Consider $f(z) = \frac{1-\cos z}{z^2}$. Then f has an isolated singularity at $z_0 = 0$. We can find a Laurent series on the annulus $0<|z|<\infty$. We have

$$\frac{1}{z^2}(1-\cos z) = \frac{1}{z^2}\left(1 - \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\right)$$

$$= -\frac{1}{z^2}\sum_{n=1}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = -\sum_{n=1}^{\infty}\frac{(-1)^n z^{2n-2}}{(2n)!}$$

$$= \frac{1}{2} - \frac{1}{4!}z^2 + \dots$$

So the principal part of f at 0 is 0. Thus, f has a removable singularity.

(2) The function $f(z) = \frac{1-\cosh z}{z^2}$ also has a removable singularity at $z_0 = 0$. The Laurent series on $0<|z|<\infty$ is

$$\frac{1}{z^2}(1-\cosh z) = \frac{1}{z^2}\left(1 - \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!}\right)$$

$$= -\sum_{n=1}^{\infty}\frac{z^{2n-2}}{(2n)!}$$

So the principal part of f at 0 is zero and so f has a removable singularity.

(3) The function $f(z) = e^{\frac{1}{z}}$ has an isolated singularity at $z_0 = 0$. The Laurent series of f on $0<|z|<\infty$ is

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{z^n \cdot n!} = 1 + \frac{1}{z\cdot 1!} + \frac{1}{z^2 \cdot 2!} + \dots +$$

So f has an essential singularity at $z_0 = 0$.

(4) The function $f(z) = \frac{1}{z^2(1-z)}$ has an isolated singularity at $z_0 = 0$. The Laurent series of f on

the annulus $0<|z|<1$ is given by

$$\frac{1}{z^2}\cdot\frac{1}{1-z} = \frac{1}{z^2}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-2}$$

$$= \underbrace{\frac{1}{z^2}+\frac{1}{z}}_{\text{principal part of } f} + \sum_{n=0}^{\infty} z^n$$

Evidently, $z_0=0$ is a pole of order $m=2$.

(5) The function $f(z)=\dfrac{z^2+z-2}{z+1}$ has an isolated singularity at $z_0=-1$. The Laurent series on the annulus $0<|z+1|<\infty$ is given by

$$\frac{z^2+z-2}{z+1} = \frac{z(z+1)-2}{z+1} = z - \frac{2}{z+1}$$

$$= -1 + z+1 - \frac{2}{z+1}.$$

Hence, $z_0=-1$ is a simple pole (a pole of order $m=1$). //

Residues at Poles

The next theorem gives a characterization of poles and an efficient method for computing the residue at a pole.

Theorem (Residue at a Pole) Let $z_0\in\mathbb{C}$ be an isolated singularity of f. Then the following are equivalent:

(a) z_0 is a pole of order m.

(b) $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ for some function ϕ that is analytic and nonzero at z_0.

Moreover, when (a) and (b) are true, the residue of f at z_0 is given by

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}}$$

Proof. ((a) $\Rightarrow$ (b)) Assume z_0 is a pole of order $m \geq 1$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

with $b_m \neq 0$ on an annulus $0 < |z-z_0| < R$. Define

$$\phi(z) = \begin{cases} (z-z_0)^m f(z), & z \neq z_0 \\ b_m, & z = z_0 \end{cases}.$$

It is clear that $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$. Moreover, $\phi(z)$ has a power series on the disk $|z-z_0| < R$:

$$\phi(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^{m+n} + b_1 (z-z_0)^{m-1} + b_2 (z-z_0)^{m-2} + \cdots + b_m$$

Hence, ϕ is analytic on the disk and hence at z_0. Moreover, $\phi(z_0) = b_m \neq 0$. So this proves (b).

((b) $\Rightarrow$ (a)) Assume $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ for some function ϕ that is analytic and nonzero at z_0. Since ϕ is analytic at z_0, there is a disk $|z-z_0| < R$ on which ϕ has a

Taylor series. Hence,

$$f(z) = \frac{1}{(z-z_0)^m} \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n$$

$$= \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m}$$

$$= \frac{\phi(z_0)}{(z-z_0)^m} + \frac{\phi^{(1)}(z_0)}{1!\,(z-z_0)^{m-1}} + \cdots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!\,(z-z_0)} + \sum_{n=0}^{\infty} \frac{\phi^{(n+m)}(z_0)\,(z-z_0)^n}{(n+m)!}.$$

Since $\phi(z_0) \neq 0$ by assumption, this proves that f has a pole of order m at z_0. Moreover, it is clear that

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

$\blacksquare$

Example

(1) The function $f(z) = \frac{z+4}{z^2+1}$ has isolated singularities at $z = i, -i$.

First, consider $z = i$. Define $\phi(z) = \frac{z+4}{z+i}$. Then clearly $f(z) = \frac{\phi(z)}{z-i}$. Moreover ϕ is analytic and nonzero at $z = i$. By the theorem, $z = i$ is a simple pole. The residue is given by

$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = \frac{i+4}{2i}$$

When $z = -i$, take $\phi(z) = \frac{z+4}{z-i}$. Then $f(z) = \frac{\phi(z)}{z+i}$, and ϕ is nonzero and analytic at $z = -i$. Hence,

$z=-i$ is a simple pole and

$$\operatorname*{Res}_{z=-i} = \phi(-i) = \frac{-i+4}{-2i}.$$

(2) The function $f(z) = \dfrac{z^3+2z}{(z-i)^3}$ has an isolated singularity at $z=i$. Consider $\phi(z) = z^3+2z$. Then $f(z) = \dfrac{\phi(z)}{(z-i)^3}$. Moreover, ϕ is analytic and nonzero at $z=i$. Hence, $z=i$ is a pole of order $m=3$. The residue is given by

$$\operatorname*{Res}_{z=i} f(z) = \frac{\phi^{(3-1)}(i)}{(3-1)!} = \frac{1}{2}(6z)\Big|_{z=i}$$
$$= 3i.$$